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ON THE EXACTNESS OF ORDINARY PARTS OVER A LOCAL FIELD OF CHARACTERISTIC p

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Abstract. Let G be a connected reductive group over a non-archimedean local field F of residue characteristic p, P be a parabolic subgroup of G, and R be a commutative ring. When R is artinian, p is nilpotent in R, and $\mathrm{char}(F) = p$, we prove that the ordinary part functor Ord_P is exact on the category of admissible smooth R-representations of G. We derive some results on Yoneda extensions between admissible smooth R-representations of G.

1. Results

Let F be a non-archimedean local field of residue characteristic p. Let $\mathbf{G}$ be a connected reductive algebraic F-group and G denote the topological group $\mathbf{G}(F)$. We let $\mathbf{P} = \mathbf{MN}$ be a parabolic subgroup of $\mathbf{G}$. We write $\bar{\mathbf{P}} = \mathbf{M}\bar{\mathbf{N}}$ for the opposite parabolic subgroup.

Let R be a commutative ring. We write $\mathrm{Mod}_G^\infty(R)$ for the category of smooth R-representations of G (i.e. $R[G]$-modules π such that for all $v \in \pi$ the stabiliser of v is open in G) and $R[G]$-linear maps. It is an R-linear abelian category. When R is noetherian, we write $\mathrm{Mod}_G^{\mathrm{adm}}(R)$ for the full subcategory of $\mathrm{Mod}_G^\infty(R)$ consisting of admissible representations (i.e. those representations π such that π^H is finitely generated over R for any open subgroup H of G). It is closed under passing to subrepresentations and extensions, thus it is an R-linear exact subcategory, but quotients of admissible representations may not be admissible when $\mathrm{char}(F) = p$ (see [AHV17, Example 4.4]).

Recall the smooth parabolic induction functor $\mathrm{Ind}_{\bar{P}}^G : \mathrm{Mod}_M^\infty(R) \to \mathrm{Mod}_G^\infty(R)$, defined on any smooth R-representation σ of M as the R-module $\mathrm{Ind}_{\bar{P}}^G(\sigma)$ of locally constant functions $f : G \to \sigma$ satisfying $f(m\bar{n}g) = m \cdot f(g)$ for all $m \in M$, $\bar{n} \in \bar{N}$, and $g \in G$, endowed with the smooth action of G by right translation. It is R-linear, exact, and commutes with small direct sums. In the other direction, there is the ordinary part functor $\mathrm{Ord}_P : \mathrm{Mod}_G^\infty(R) \to \mathrm{Mod}_M^\infty(R)$ ([Eme10a, Vig16]). It is R-linear and left exact. When R is noetherian, Ord_P also commutes with small inductive limits, both functors respect admissibility, and the restriction of Ord_P to $\mathrm{Mod}_G^{\mathrm{adm}}(R)$ is right adjoint to the restriction of $\mathrm{Ind}_{\bar{P}}^G$ to $\mathrm{Mod}_M^{\mathrm{adm}}(R)$.

Theorem 1. *If R is artinian, p is nilpotent in R, and $\mathrm{char}(F) = p$, then Ord_P is exact on $\mathrm{Mod}_G^{\mathrm{adm}}(R)$.*

Thus the situation is very different from the case $\mathrm{char}(F) = 0$ (see [Eme10b]). On the other hand if R is artinian and p is invertible in R, then Ord_P is isomorphic on $\mathrm{Mod}_G^{\mathrm{adm}}(R)$ to the Jacquet functor with respect to P (i.e. the N-coinvariants) twisted by the inverse of the modulus character δ_P of P ([AHV17, Corollary 4.19]), so that it is exact on $\mathrm{Mod}_G^{\mathrm{adm}}(R)$ without any assumption on $\mathrm{char}(F)$.

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Remark. Without any assumption on R, $\mathrm{Ind}_P^G : \mathrm{Mod}_M^\infty(R) \to \mathrm{Mod}_G^\infty(R)$ admits a left adjoint $\mathrm{L}_P^G : \mathrm{Mod}_G^\infty(R) \to \mathrm{Mod}_M^\infty(R)$ (the Jacquet functor with respect to P) and a right adjoint $\mathrm{R}_P^G : \mathrm{Mod}_G^\infty(R) \to \mathrm{Mod}_M^\infty(R)$ ([Vig16, Proposition 4.2]). If R is noetherian and p is nilpotent in R, then R_P^G is isomorphic to $\mathrm{Ord}_{\bar{P}}$ on $\mathrm{Mod}_G^{\mathrm{adm}}(R)$ ([AHV17, Corollary 4.13]). Thus under the assumptions of Theorem 1, R_P^G is exact on $\mathrm{Mod}_G^{\mathrm{adm}}(R)$. On the other hand if R is noetherian and p is invertible in R, then R_P^G is expected to be isomorphic to $\delta_P \, \mathrm{L}_{\bar{P}}^G$ ('second adjointness'), and this is proved in the following cases: when R is the field of complex numbers ([Ber87]) or an algebraically closed field of characteristic $\ell \neq p$ ([Vig96, II.3.8 2)]); when $\mathbf{G}$ is a Levi subgroup of a general linear group or a classical group with $p \neq 2$ ([Dat09, Théorème 1.5]); when $\mathbf{P}$ is a minimal parabolic subgroup of $\mathbf{G}$ (see also [Dat09]). In particular, L_P^G and R_P^G are exact in all these cases.

Question. Are L_P^G and R_P^G exact when R is noetherian, p is nilpotent in R, and $\mathrm{char}(F) = p$?

We derive from Theorem 1 some results on Yoneda extensions between admissible R-representations of G. We compute the R-modules $\mathrm{Ext}_G^\bullet$ in $\mathrm{Mod}_G^{\mathrm{adm}}(R)$.

Corollary 2. *Assume R artinian, p nilpotent in R, and* $\mathrm{char}(F) = p$. *Let σ and π be admissible R-representations of M and G respectively. For all $n \geq 0$, there is a natural R-linear isomorphism*

$$\mathrm{Ext}_M^n(\sigma, \mathrm{Ord}_P(\pi)) \xrightarrow{\sim} \mathrm{Ext}_G^n(\mathrm{Ind}_{\bar{P}}^G(\sigma), \pi).$$

This is in contrast with the case $\mathrm{char}(F) = 0$ (see [Hau16b]). A direct consequence of Corollary 2 is that under the same assumptions, $\mathrm{Ind}_{\bar{P}}^G$ induces an isomorphism between the Ext^n for all $n \geq 0$ (Corollary 5). When $R = C$ is an algebraically closed field of characteristic p and $\mathrm{char}(F) = p$, we determine the extensions between certain irreducible admissible C-representations of G using the classification of [AHHV17] (Proposition 6). In particular, we prove that there exists no non-split extension of an irreducible admissible C-representation π of G by a supersingular C-representation of G when π is not the extension to G of a supersingular representation of a Levi subgroup of G (Corollary 7). When $\mathbf{G} = \mathrm{GL}_2$, this was first proved by Hu ([Hu17, Theorem A.2]).

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2. Proofs

2.1. **Hecke action.** In this subsection, $\mathbf{M}$ denotes a linear algebraic F-group and $\mathbf{N}$ denotes a split unipotent algebraic F-group (see [CGP15, Appendix B]) endowed with an action of $\mathbf{M}$ that we identify with the conjugation in $\mathbf{M} \ltimes \mathbf{N}$. We fix an open submonoid M^+ of M and a compact open subgroup N_0 of N stable under conjugation by M^+.

If π is a smooth R-representation of $M^+ \ltimes N_0$, then the R-modules $\mathrm{H}^\bullet(N_0, \pi)$, computed using the homogeneous cochain complex $\mathrm{C}^\bullet(N_0, \pi)$ (see [NSW08, § I.2]), are naturally endowed with the Hecke action of M^+, defined as the composite

$$\mathrm{H}^\bullet(N_0, \pi) \xrightarrow{m} \mathrm{H}^\bullet(mN_0m^{-1}, \pi) \xrightarrow{\mathrm{cor}} \mathrm{H}^\bullet(N_0, \pi)$$

for all $m \in M^+$. At the level of cochains, this action is explicitly given as follows (see [NSW08, § I.5]). We fix a set of representatives $\overline{N_0/mN_0m^{-1}} \subseteq N_0$ of the left cosets N_0/mN_0m^{-1} and we write $n \mapsto \bar{n}$ for the projection $N_0 \twoheadrightarrow \overline{N_0/mN_0m^{-1}}$.

For $\phi \in \mathrm{C}^k(N_0, \pi)$, we have

$$(1) \quad (m \cdot \phi)(n_0, \dots, n_k) \\ = \sum_{\bar{n} \in \overline{N_0/mN_0m^{-1}}} \bar{n}m \cdot \phi(m^{-1}\bar{n}^{-1}n_0\overline{n_0^{-1}\bar{n}}m, \dots, m^{-1}\bar{n}^{-1}n_k\overline{n_k^{-1}\bar{n}}m)$$

for all $(n_0, \dots, n_k) \in N_0^{k+1}$.

Lemma 3. *Assume p nilpotent in R and $\mathrm{char}(F) = p$. Let π be a smooth R-representation of $M^+ \ltimes N_0$ and $m \in M^+$. If the Hecke action $h_{N_0,m}$ of m on π^{N_0} is locally nilpotent (i.e. for all $v \in \pi^{N_0}$ there exists $r \geq 0$ such that $h_{N_0,m}^r(v) = 0$), then the Hecke action of m on $\mathrm{H}^k(N_0, \pi)$ is locally nilpotent for all $k \geq 0$.*

Proof. First, we prove the lemma when $pR = 0$, i.e. R is a commutative $\mathbb{F}_p$-algebra. We assume that the Hecke action of m on π^{N_0} is locally nilpotent and we prove the result together with the following fact: there exists a set of representatives $\overline{N_0/mN_0m^{-1}} \subseteq N_0$ of the left cosets N_0/mN_0m^{-1} such that the action of

$$S \coloneqq \sum_{\bar{n} \in \overline{N_0/mN_0m^{-1}}} \bar{n}m \in \mathbb{F}_p[M^+ \ltimes N_0]$$

on π is locally nilpotent.

We proceed by induction on the dimension of $\mathbf{N}$ (recall that $\mathbf{N}$ is split so that it is smooth and connected). If $\mathbf{N} = 1$, then the (Hecke) action of m on $\pi^{N_0} = \pi$ is locally nilpotent by assumption, so that the result and the fact are trivially true. Assume $\mathbf{N} \neq 1$ and that the result and the fact are true for groups of smaller dimension. Since $\mathbf{N}$ is split, it admits a non-trivial central subgroup isomorphic to the additive group. We let $\mathbf{N}'$ be the subgroup of $\mathbf{N}$ generated by all such subgroups. It is a non-trivial vector group (i.e. isomorphic to a direct product of copies of the additive group) which is central (hence normal) in $\mathbf{N}$ and stable under conjugation by $\mathbf{M}$ (since it is a characteristic subgroup of $\mathbf{N}$). We set $\mathbf{N}'' \coloneqq \mathbf{N}/\mathbf{N}'$. It is a split unipotent algebraic F-group endowed with the induced action of $\mathbf{M}$ and $\dim(\mathbf{N}'') < \dim(\mathbf{N})$. Since $\mathbf{N}'$ is split, we have $N'' = N/N'$. We write N_0' and N_0'' for the compact open subgroups $N' \cap N_0$ and N_0/N_0' of N' and N'' respectively. They are stable under conjugation by M^+. We fix a set-theoretic section $[-] : N_0'' \hookrightarrow N_0$.

Since $\mathbf{N}'$ is commutative and p-torsion, N_0' is a compact $\mathbb{F}_p$-vector space. Thus for any open subgroup N_1' of N_0', the short exact sequence of compact $\mathbb{F}_p$-vector spaces

$$0 \to N_1' \to N_0' \to N_0'/N_1' \to 0$$

splits. Indeed, it admits an $\mathbb{F}_p$-linear splitting (since $\mathbb{F}_p$ is a field) which is automatically continuous (since N_0'/N_1' is discrete). In particular with $N_1' = mN_0'm^{-1}$, we may and do fix a section $N_0'/mN_0'm^{-1} \hookrightarrow N_0'$. We write $\overline{N_0'/mN_0'm^{-1}}$ for its image, so that $N_0' = \overline{N_0'/mN_0'm^{-1}} \times mN_0'm^{-1}$, and $n' \mapsto \bar{n}'$ for the projection $N_0' \twoheadrightarrow \overline{N_0'/mN_0'm^{-1}}$. We set

$$S' \coloneqq \sum_{\bar{n}' \in \overline{N_0'/mN_0'm^{-1}}} \bar{n}'m \in \mathbb{F}_p[M^+ \ltimes N_0'].$$

For all $n_0' \in N_0'$, we have $n_0' = \bar{n}_0'(\bar{n}_0'^{-1}n_0')$ with $\bar{n}_0'^{-1}n_0' \in mN_0'm^{-1}$, thus

$$n_0'S' = \sum_{\bar{n}' \in \overline{N_0'/mN_0'm^{-1}}} (\bar{n}_0'\bar{n}')m(m^{-1}(\bar{n}_0'^{-1}n_0')m) = S'(m^{-1}(\bar{n}_0'^{-1}n_0')m)$$

with $m^{-1}(\bar{n}_0'^{-1}n_0')m \in N_0'$ (in the first equality we use the fact that N_0' is commutative and in the second one we use the fact that $\overline{N_0'/mN_0'm^{-1}}$ is a group). Therefore, there is an inclusion $\mathbb{F}_p[N_0']S' \subseteq S'\mathbb{F}_p[N_0']$.

The R-module $\pi^{N_0'}$, endowed with the induced action of N_0'' and the Hecke action of M^+ with respect to N_0', is a smooth R-representation of $M^+ \ltimes N_0''$ (see the proof of [Hau16a, Lemme 3.2.1] in degree 0). On $\pi^{N_0'}$, the Hecke action of m with respect to N_0' coincides with the action of S' by definition. On $(\pi^{N_0'})^{N_0''} = \pi^{N_0}$, the Hecke action of m with respect to N_0'' coincides with the Hecke action of m with respect to N_0 (see the proof of [Hau16a, Lemme 3.2.2]) which is locally nilpotent by assumption. Thus by the induction hypothesis, there exists a set of representatives $\overline{N_0''/mN_0''m^{-1}} \subseteq N_0''$ of the left cosets $N_0''/mN_0''m^{-1}$ such that the action of

$$S \coloneqq \sum_{\bar{n}'' \in \overline{N_0''/mN_0''m^{-1}}} [\bar{n}'']S' \in \mathbb{F}_p[M^+ \ltimes N_0]$$

on $\pi^{N_0'}$ is locally nilpotent. Moreover, there is an inclusion $\mathbb{F}_p[N_0']S \subseteq S\mathbb{F}_p[N_0']$ (because N_0' is central in N_0 and $\mathbb{F}_p[N_0']S' \subseteq S'\mathbb{F}_p[N_0']$).

We prove the fact. By [Hau16c, Lemme 2.1],

$$\overline{N_0/mN_0m^{-1}} \coloneqq \{[\bar{n}'']\bar{n}' : \bar{n}'' \in \overline{N_0''/mN_0''m^{-1}}, \bar{n}' \in \overline{N_0'/mN_0'm^{-1}}\} \subseteq N_0$$

is a set of representatives of the left cosets N_0/mN_0m^{-1}, and by definition

$$S = \sum_{\bar{n} \in \overline{N_0/mN_0m^{-1}}} \bar{n}m.$$

We prove that the action of S on π is locally nilpotent. We proceed as in the proof of [Hu12, Théorème 5.1 (i)]. Let $v \in \pi$ and set $\pi_r \coloneqq \mathbb{F}_p[N_0'] \cdot (S^r \cdot v)$ for all $r \geq 0$. Since $\mathbb{F}_p[N_0']S \subseteq S\mathbb{F}_p[N_0']$, we have $\pi_{r+1} \subseteq S \cdot \pi_r$ for all $r \geq 0$. Since N_0' is compact, we have $\dim_{\mathbb{F}_p}(\pi_r) < \infty$ for all $r \geq 0$. If $S^r \cdot v \neq 0$, i.e. $\pi_r \neq 0$, for some $r \geq 0$, then $\pi_r^{N_0'} \neq 0$ (because N_0' is a pro-p group and π_r is a non-zero $\mathbb{F}_p$-vector space) so that $\dim_{\mathbb{F}_p}(S \cdot \pi_r) < \dim_{\mathbb{F}_p} \pi_r$ (because the action of S on $\pi^{N_0'}$ is locally nilpotent). Therefore $\pi_r = 0$, i.e. $S^r \cdot v = 0$, for all $r \geq \dim_{\mathbb{F}_p}(\pi_0)$.

We prove the result. The R-modules $\mathrm{H}^\bullet(N_0', \pi)$, endowed with the induced action of N_0'' and the Hecke action of M^+, are smooth R-representations of $M^+ \ltimes N_0''$ (see the proof of [Hau16a, Lemme 3.2.1][1]). At the level of cochains, the actions of $n'' \in N_0''$ and m are explicitly given as follows. For $\phi \in \mathrm{C}^j(N_0', \pi)$, we have

$$(n'' \cdot \phi)(n_0', \dots, n_j') = [n''] \cdot \phi(n_0', \dots, n_j') \tag{2}$$

$$(m \cdot \phi)(n_0', \dots, n_j') = S' \cdot \phi(m^{-1}n_0'\bar{n}_0'^{-1}m, \dots, m^{-1}n_j'\bar{n}_j'^{-1}m) \tag{3}$$

for all $(n_0', \dots, n_j') \in N_0'^{j+1}$ (for (2) we use the fact that N_0' is central in N_0, for (3) we use (1) and the fact that $n' \mapsto \bar{n}'$ is a group homomorphism $N_0' \to \overline{N_0'/mN_0'm^{-1}}$). Using (2) and (3), we can give explicitly the Hecke action of m on $\mathrm{H}^\bullet(N_0', \pi)^{N_0''}$ at the level of cochains as follows. For $\phi \in \mathrm{C}^j(N_0', \pi)$, we have

$$(m \cdot \phi)(n_0', \dots, n_j') = S \cdot \phi(m^{-1}n_0'\bar{n}_0'^{-1}m, \dots, m^{-1}n_j'\bar{n}_j'^{-1}m)$$

for all $(n_0', \dots, n_j') \in N_0'^{j+1}$. Since the action of S on π is locally nilpotent and the image of a locally constant cochain is finite by compactness of N_0', we deduce that the Hecke action of m on $\mathrm{H}^j(N_0', \pi)^{N_0''}$ is locally nilpotent for all $j \geq 0$. Thus the Hecke action of m on $\mathrm{H}^i(N_0'', \mathrm{H}^j(N_0', \pi))$ is locally nilpotent for all $i, j \geq 0$ by the induction hypothesis. We conclude using the spectral sequence of smooth R-representations of M^+

$$\mathrm{H}^i(N_0'', \mathrm{H}^j(N_0', \pi)) \Rightarrow \mathrm{H}^{i+j}(N_0, \pi)$$

(see the proof of [Hau16a, Proposition 3.2.3] and footnote 1).

[1]We do not know whether [Eme10b, Proposition 2.1.11] holds true when $\mathrm{char}(F) = p$, but [Hau16a, Lemme 3.1.1] does and any injective object of $\mathrm{Mod}^\infty_{M^+ \ltimes N_0}(R)$ is still N_0-acyclic.

Now, we prove the lemma without assuming $pR = 0$. We proceed by induction on the degree of nilpotency r of p in R. If $r \leq 1$, then the lemma is already proved. We assume $r > 1$ and that we know the lemma for rings in which the degree of nilpotency of p is $r-1$. There is a short exact sequence of smooth R-representations of $M^+ \ltimes N_0$

$$0 \to p\pi \to \pi \to \pi/p\pi \to 0.$$

Taking the N_0-cohomology yields a long exact sequence of smooth R-representations of M^+

$$0 \to (p\pi)^{N_0} \to \pi^{N_0} \to (\pi/p\pi)^{N_0} \to \mathrm{H}^1(N_0, p\pi) \to \cdots . \tag{4}$$

If the Hecke action of m on π^{N_0} is locally nilpotent, then the Hecke action of m on $(p\pi)^{N_0}$ is also locally nilpotent so that the Hecke action of m on $\mathrm{H}^k(N_0, p\pi)$ is locally nilpotent for all $k \geq 0$ by the induction hypothesis (since $p\pi$ is an $R/p^{r-1}R$-module). Using (4), we deduce that the Hecke action of m on $(\pi/p\pi)^{N_0}$ is also locally nilpotent so that the Hecke action of m on $\mathrm{H}^k(N_0, \pi/p\pi)$ is locally nilpotent for all $k \geq 0$ (since $\pi/p\pi$ is an $\mathbb{F}_p$-vector space). Using again (4), we conclude that the Hecke action of m on $\mathrm{H}^k(N_0, \pi)$ is locally nilpotent for all $k \geq 0$. ☐

2.2. **Proof of the main result.** We fix a compact open subgroup N_0 of N and we let M^+ be the open submonoid of M consisting of those elements m contracting N_0 (i.e. $mN_0m^{-1} \subseteq N_0$). We let $\mathbf{Z_M}$ denote the centre of $\mathbf{M}$ and we set $Z_M^+ \coloneqq Z_M \cap M^+$. We fix an element $z \in Z_M^+$ strictly contracting N_0 (i.e. $\cap_{r \geq 0} z^r N_0 z^{-r} = 1$).

Recall that the ordinary part of a smooth R-representation π of P is the smooth R-representation of M

$$\mathrm{Ord}_P(\pi) \coloneqq (\mathrm{Ind}_{M^+}^M(\pi^{N_0}))^{Z_M-\mathrm{l.fin}}$$

where $\mathrm{Ind}_{M^+}^M(\pi^{N_0})$ is defined as the R-module of functions $f : M \to \pi^{N_0}$ such that $f(mm') = m \cdot f(m')$ for all $m \in M^+$ and $m' \in M$, endowed with the action of M by right translation, and the superscript ${}^{Z_M-\mathrm{l.fin}}$ denotes the subrepresentation consisting of locally Z_M-finite elements (i.e. those elements f such that $R[Z_M] \cdot f$ is contained in a finitely generated R-submodule). The action of M on the latter is smooth by [Vig16, Remark 7.6]. If R is artinian and π^{N_0} is locally Z_M^+-finite (i.e. it may be written as the union of finitely generated Z_M^+-invariant R-submodules), then there is a natural R-linear isomorphism

$$\mathrm{Ord}_P(\pi) \xrightarrow{\sim} R[z^{\pm 1}] \otimes_{R[z]} \pi^{N_0} \tag{5}$$

(cf. [Eme10b, Lemma 3.2.1 (1)], whose proof also works when $\mathrm{char}(F) = p$ and over any artinian ring).

If σ is a smooth R-representation of M, then the R-module $\mathcal{C}_c^\infty(N, \sigma)$ of locally constant functions $f : N \to \sigma$ with compact support, endowed with the action of N by right translation and the action of M given by $(m \cdot f) : n \mapsto m \cdot f(m^{-1}nm)$ for all $m \in M$, is a smooth R-representation of P. Thus we obtain a functor $\mathcal{C}_c^\infty(N, -) : \mathrm{Mod}_M^\infty(R) \to \mathrm{Mod}_P^\infty(R)$. It is R-linear, exact, and commutes with small direct sums. The results of [Eme10a, § 4.2] hold true when $\mathrm{char}(F) = p$ and over any ring, thus the functors

$$\mathcal{C}_c^\infty(N, -) : \mathrm{Mod}_M^\infty(R)^{Z_M-\mathrm{l.fin}} \to \mathrm{Mod}_P^\infty(R)$$
$$\mathrm{Ord}_P : \mathrm{Mod}_P^\infty(R) \to \mathrm{Mod}_M^\infty(R)^{Z_M-\mathrm{l.fin}}$$

are adjoint and the unit of the adjunction is an isomorphism.

Lemma 4. *Assume R artinian, p nilpotent in R, and $\mathrm{char}(F) = p$. Let π be a smooth R-representation of P. If π^{N_0} is locally Z_M^+-finite, then the Hecke action of z on $\mathrm{H}^k(N_0, \pi)$ is locally nilpotent for all $k \geq 1$.*

Proof. We set $\sigma := \mathrm{Ord}_P(\pi)$. The counit of the adjunction between $\mathcal{C}_c^\infty(N,-)$ and Ord_P induces a natural morphism of smooth R-representations of P

$$\mathcal{C}_c^\infty(N,\sigma) \to \pi. \tag{6}$$

Taking the N_0-invariants yields a morphism of smooth R-representations of M^+

$$\mathcal{C}_c^\infty(N,\sigma)^{N_0} \to \pi^{N_0}. \tag{7}$$

By definition, σ is locally Z_M-finite so it may be written as the union of finitely generated Z_M-invariant R-submodules $(\sigma_i)_{i\in I}$. Thus $\mathcal{C}_c^\infty(N,\sigma)^{N_0}$ is the union of the finitely generated Z_M^+-invariant R-submodules $(\mathcal{C}^\infty(z^{-r}N_0z^r,\sigma_i)^{N_0})_{r\geq 0,i\in I}$, so it is locally Z_M^+-finite. By assumption, π^{N_0} is also locally Z_M^+-finite. Therefore, using (5) and its analogue with $\mathcal{C}_c^\infty(N,\sigma)$ instead of π, the localisation with respect to z of (7) is the natural morphism of smooth R-representations of M

$$\mathrm{Ord}_P(\mathcal{C}_c^\infty(N,\sigma)) \to \mathrm{Ord}_P(\pi)$$

induced by applying the functor Ord_P to (6), and it is an isomorphism since the unit of the adjunction between $\mathcal{C}_c^\infty(N,-)$ and Ord_P is an isomorphism.

Let κ (resp. ι) be the kernel (resp. image) of (6), hence two short exact sequences of smooth R-representations of P

$$0 \to \kappa \to \mathcal{C}_c^\infty(N,\sigma) \to \iota \to 0 \tag{8}$$
$$0 \to \iota \to \pi \to \pi/\iota \to 0 \tag{9}$$

such that the third arrow of (8) and the second arrow of (9) fit into a commutative diagram of smooth R-representations of P

$$\begin{tikzcd} \mathcal{C}_c^\infty(N,\sigma) \arrow[rr] \arrow[dr, two heads] & & \pi \\ & \iota \arrow[ur, hook] & \end{tikzcd}$$

whose upper arrow is (6). Taking the N_0-invariants yields a commutative diagram of smooth R-representations of M^+

$$\begin{tikzcd} \mathcal{C}_c^\infty(N,\sigma)^{N_0} \arrow[rr] \arrow[dr] & & \pi^{N_0} \\ & \iota^{N_0} \arrow[ur, hook] & \end{tikzcd}$$

whose upper arrow is (7). Since the localisation with respect to z of the latter is an isomorphism, the localisation with respect to z of the injection $\iota^{N_0} \hookrightarrow \pi^{N_0}$ is surjective, thus it is an isomorphism (as it is also injective by exactness of localisation). Therefore the localisation with respect to z of the morphism $\mathcal{C}_c^\infty(N,\sigma)^{N_0} \to \iota^{N_0}$ is an isomorphism.

Since $\mathcal{C}_c^\infty(N,\sigma) \cong \bigoplus_{n\in N/N_0} \mathcal{C}^\infty(nN_0,\sigma)$ as a smooth R-representation of N_0, it is N_0-acyclic (see [NSW08, § I.3]). Thus the long exact sequence of N_0-cohomology induced by (8) yields an exact sequence of smooth R-representations of M^+

$$0 \to \kappa^{N_0} \to \mathcal{C}_c^\infty(N,\sigma)^{N_0} \to \iota^{N_0} \to \mathrm{H}^1(N_0,\kappa) \to 0 \tag{10}$$

and an isomorphism of smooth R-representations of M^+

$$\mathrm{H}^k(N_0,\iota) \xrightarrow{\sim} \mathrm{H}^{k+1}(N_0,\kappa) \tag{11}$$

for all $k \geq 1$. Since the localisation with respect to z of the third arrow of (10) is an isomorphism, the Hecke action of z on κ^{N_0} is locally nilpotent. Thus the Hecke action of z on $\mathrm{H}^k(N_0,\kappa)$ is locally nilpotent for all $k \geq 0$ by Lemma 3. Using (11), we deduce that the Hecke action of z on $\mathrm{H}^k(N_0,\iota)$ is locally nilpotent for all $k \geq 1$.

Taking the N_0-cohomology of (9) yields a long exact sequence of smooth R-representations of M^+

$$0 \to \iota^{N_0} \to \pi^{N_0} \to (\pi/\iota)^{N_0} \to \mathrm{H}^1(N_0, \iota) \to \cdots . \tag{12}$$

Since the localisation with respect to z of the second arrow is an isomorphism and the Hecke action of z on $\mathrm{H}^1(N_0, \iota)$ is locally nilpotent, the Hecke action of z on $(\pi/\iota)^{N_0}$ is locally nilpotent. Thus the Hecke action of z on $\mathrm{H}^k(N_0, \pi/\iota)$ is locally nilpotent for all $k \geq 0$ by Lemma 3. We conclude using (12) and the fact that the Hecke action of z on $\mathrm{H}^k(N_0, \iota)$ is locally nilpotent for all $k \geq 1$. □

Proof of Theorem 1. Assume R artinian, p nilpotent in R, and $\mathrm{char}(F) = p$. Let

$$0 \to \pi_1 \to \pi_2 \to \pi_3 \to 0 \tag{13}$$

be a short exact sequence of admissible R-representations of G. Taking the N_0-invariants yields an exact sequence of smooth R-representations of M^+

$$0 \to \pi_1^{N_0} \to \pi_2^{N_0} \to \pi_3^{N_0} \to \mathrm{H}^1(N_0, \pi_1). \tag{14}$$

The terms $\pi_1^{N_0}, \pi_2^{N_0}, \pi_3^{N_0}$ are locally Z_M^+-finite (cf. [Eme10b, Theorem 3.4.7 (1)], whose proof in degree 0 also works when $\mathrm{char}(F) = p$ and over any noetherian ring) and the Hecke action of z on $\mathrm{H}^1(N_0, \pi_1)$ is locally nilpotent by Lemma 4. Therefore, using (5), the localisation with respect to z of (14) is the short sequence of admissible R-representations of M

$$0 \to \mathrm{Ord}_P(\pi_1) \to \mathrm{Ord}_P(\pi_2) \to \mathrm{Ord}_P(\pi_3) \to 0$$

induced by applying the functor Ord_P to (13), and it is exact by exactness of localisation. □

2.3. **Results on extensions.** We assume R noetherian. The R-linear category $\mathrm{Mod}_G^{\mathrm{adm}}(R)$ is not abelian in general, but merely exact in the sense of Quillen ([Qui73]). An exact sequence of admissible R-representations of G is an exact sequence of smooth R-representations of G

$$\cdots \to \pi_{n-1} \to \pi_n \to \pi_{n+1} \to \cdots$$

such that the kernel and the cokernel of every arrow are admissible. In particular, each term of the sequence is also admissible.

For $n \geq 0$ and π, π' two admissible R-representations of G, we let $\mathrm{Ext}_G^n(\pi', \pi)$ denote the R-module of n-fold Yoneda extensions ([Yon60]) of π' by π in $\mathrm{Mod}_G^{\mathrm{adm}}(R)$, defined as equivalence classes of exact sequences

$$0 \to \pi \to \pi_1 \to \cdots \to \pi_n \to \pi' \to 0.$$

We let $D(G)$ denote the derived category of $\mathrm{Mod}_G^{\mathrm{adm}}(R)$ ([Nee90, Kel96, Büh10]). The results of [Ver96, § III.3.2] on the Yoneda construction carry over to this setting (see e.g. [Pos11, Proposition A.13]), hence a natural R-linear isomorphism

$$\mathrm{Ext}_G^n(\pi', \pi) \cong \mathrm{Hom}_{D(G)}(\pi', \pi[n]).$$

Proof of Corollary 2. Since Ind_P^G and Ord_P are exact adjoint functors between $\mathrm{Mod}_M^{\mathrm{adm}}(R)$ and $\mathrm{Mod}_G^{\mathrm{adm}}(R)$ by Theorem 1, they induce adjoint functors between $D(M)$ and $D(G)$, hence natural R-linear isomorphisms

$$\begin{aligned}\mathrm{Ext}_M^n(\sigma, \mathrm{Ord}_P(\pi)) &\cong \mathrm{Hom}_{D(M)}(\sigma, \mathrm{Ord}_P(\pi)[n]) \\ &\cong \mathrm{Hom}_{D(G)}(\mathrm{Ind}_P^G(\sigma), \pi[n]) \\ &\cong \mathrm{Ext}_G^n(\mathrm{Ind}_P^G(\sigma), \pi)\end{aligned}$$

for all $n \geq 0$. □

Remark. We give a more explicit proof of Corollary 2. The exact functor $\mathrm{Ind}_{\bar{P}}^{G}$ and the counit of the adjunction between $\mathrm{Ind}_{\bar{P}}^{G}$ and Ord_P induce an R-linear morphism

$$\mathrm{Ext}_M^n(\sigma, \mathrm{Ord}_P(\pi)) \to \mathrm{Ext}_G^n(\mathrm{Ind}_{\bar{P}}^{G}(\sigma), \pi). \tag{15}$$

In the other direction, the exact (by Theorem 1) functor Ord_P and the unit of the adjunction between $\mathrm{Ind}_{\bar{P}}^{G}$ and Ord_P induce an R-linear morphism

$$\mathrm{Ext}_G^n(\mathrm{Ind}_{\bar{P}}^{G}(\sigma), \pi) \to \mathrm{Ext}_M^n(\sigma, \mathrm{Ord}_P(\pi)). \tag{16}$$

We prove that (16) is the inverse of (15). For $n = 0$ this is the unit-counit equations. Assume $n \geq 1$ and let

$$0 \to \mathrm{Ord}_P(\pi) \to \sigma_1 \to \cdots \to \sigma_n \to \sigma \to 0 \tag{17}$$

be an exact sequence of admissible R-representations of M. By [Yon60, § 3], the image of the class of (17) under (15) is the class of any exact sequence of admissible R-representations of G

$$0 \to \pi \to \pi_1 \to \cdots \to \pi_n \to \mathrm{Ind}_{\bar{P}}^{G}(\sigma) \to 0 \tag{18}$$

such that there exists a commutative diagram of admissible R-representations of G

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ind}_{\bar{P}}^{G}(\mathrm{Ord}_P(\pi)) \arrow[r] \arrow[d] & \mathrm{Ind}_{\bar{P}}^{G}(\sigma_1) \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathrm{Ind}_{\bar{P}}^{G}(\sigma_n) \arrow[r] \arrow[d] & \mathrm{Ind}_{\bar{P}}^{G}(\sigma) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi \arrow[r] & \pi_1 \arrow[r] & \cdots \arrow[r] & \pi_n \arrow[r] & \mathrm{Ind}_{\bar{P}}^{G}(\sigma) \arrow[r] & 0
\end{tikzcd}$$

in which the upper row is obtained from (17) by applying the exact functor $\mathrm{Ind}_{\bar{P}}^{G}$, the lower row is (18), and the leftmost vertical arrow is the natural morphism induced by the counit of the adjunction between $\mathrm{Ind}_{\bar{P}}^{G}$ and Ord_P. Applying the exact functor Ord_P to the diagram and using the unit of the adjunction between $\mathrm{Ind}_{\bar{P}}^{G}$ and Ord_P yields a commutative diagram of admissible R-representations of M

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ord}_P(\pi) \arrow[r] \arrow[d, equal] & \sigma_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & \sigma_n \arrow[r] \arrow[d] & \sigma \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ord}_P(\pi) \arrow[r] & \mathrm{Ord}_P(\pi_1) \arrow[r] & \cdots \arrow[r] & \mathrm{Ord}_P(\pi_n) \arrow[r] & \mathrm{Ord}_P(\mathrm{Ind}_{\bar{P}}^{G}(\sigma)) \arrow[r] & 0
\end{tikzcd}$$

in which the lower row is obtained from (18) by applying the exact functor Ord_P, the upper row is (17), and the rightmost vertical arrow is the natural morphism induced by the unit of the adjunction between $\mathrm{Ind}_{\bar{P}}^{G}$ and Ord_P. The leftmost vertical morphism is the identity by the unit-counit equations. Thus the image of the class of (18) under (16) is the class of (17) by [Yon60, § 3]. We have proved that (16) is a left inverse of (15). The proof that it is a right inverse is dual.

Corollary 5. *Assume R artinian, p nilpotent in R, and $\mathrm{char}(F) = p$. Let σ and σ' be two admissible R-representations of M. The functor $\mathrm{Ind}_{\bar{P}}^{G}$ induces an R-linear isomorphism*

$$\mathrm{Ext}_M^n(\sigma', \sigma) \xrightarrow{\sim} \mathrm{Ext}_G^n(\mathrm{Ind}_{\bar{P}}^{G}(\sigma'), \mathrm{Ind}_{\bar{P}}^{G}(\sigma))$$

for all $n \geq 0$.

Proof. The isomorphism in the statement is the composite

$$\mathrm{Ext}_M^n(\sigma', \sigma) \xrightarrow{\sim} \mathrm{Ext}_M^n(\sigma', \mathrm{Ord}_P(\mathrm{Ind}_{\bar{P}}^{G}(\sigma))) \xrightarrow{\sim} \mathrm{Ext}_G^n(\mathrm{Ind}_{\bar{P}}^{G}(\sigma'), \mathrm{Ind}_{\bar{P}}^{G}(\sigma))$$

where the first isomorphism is induced by the unit of the adjunction between $\mathrm{Ind}_{\bar{P}}^{G}$ and Ord_P, which is an isomorphism, and the second one is the isomorphism of Corollary 2 with σ' and $\mathrm{Ind}_{\bar{P}}^{G}(\sigma)$ instead of σ and π respectively. ∎

We fix a minimal parabolic subgroup $\mathbf{B} \subseteq \mathbf{G}$, a maximal split torus $\mathbf{S} \subseteq \mathbf{B}$, and we write Δ for the set of simple roots of $\mathbf{S}$ in $\mathbf{B}$. We say that a parabolic subgroup $\mathbf{P} = \mathbf{MN}$ of $\mathbf{G}$ is *standard* if $\mathbf{B} \subseteq \mathbf{P}$ and $\mathbf{S} \subseteq \mathbf{M}$. In this case, we write $\Delta_{\mathbf{P}}$ for the corresponding subset of Δ, and given $\alpha \in \Delta_{\mathbf{P}}$ (resp. $\alpha \in \Delta \backslash \Delta_{\mathbf{P}}$) we write $\mathbf{P}^\alpha = \mathbf{M}^\alpha \mathbf{N}^\alpha$ (resp. $\mathbf{P}_\alpha = \mathbf{M}_\alpha \mathbf{N}_\alpha$) for the standard parabolic subgroup corresponding to $\Delta_{\mathbf{P}} \backslash \{\alpha\}$ (resp. $\Delta_{\mathbf{P}} \sqcup \{\alpha\}$).

Let C be an algebraically closed field of characteristic p. Given a standard parabolic subgroup $P = MN$ and a smooth C-representation σ of M, there exists a largest standard parabolic subgroup $P(\sigma) = M(\sigma)N(\sigma)$ such that the inflation of σ to P extends to a smooth C-representation ${}^{\mathrm{e}}\sigma$ of $P(\sigma)$, and this extension is unique ([AHHV17, II.7 Corollary 1]). We say that a smooth C-representation of G is *supercuspidal* if it is irreducible, admissible, and does not appear as a subquotient of $\mathrm{Ind}_P^G(\sigma)$ for any proper parabolic subgroup $P = MN$ of G and any irreducible admissible C-representation σ of M. A *supercuspidal standard* $C[G]$*-triple* is a triple (P, σ, Q) where $P = MN$ is a standard parabolic subgroup, σ is a supercuspidal C-representation of M, and Q is a parabolic subgroup of G such that $P \subseteq Q \subseteq P(\sigma)$. To such a triple is attached in [AHHV17] a smooth C-representation of G

$$\mathrm{I}_G(P, \sigma, Q) \coloneqq \mathrm{Ind}_{P(\sigma)}^G({}^{\mathrm{e}}\sigma \otimes \mathrm{St}_Q^{P(\sigma)})$$

where $\mathrm{St}_Q^{P(\sigma)} \coloneqq \mathrm{Ind}_Q^{P(\sigma)}(1) / \sum_{Q \subsetneq Q' \subseteq P(\sigma)} \mathrm{Ind}_{Q'}^{P(\sigma)}(1)$ (here 1 denotes the trivial C-representation) is the inflation to $P(\sigma)$ of the generalised Steinberg representation of $M(\sigma)$ with respect to $M(\sigma) \cap Q$ ([GK14, Ly15]). It is irreducible and admissible ([AHHV17, I.3 Theorem 1]).

Proposition 6. *Assume* $\mathrm{char}(F) = p$. *Let* (P, σ, Q) *and* (P', σ', Q') *be two supercuspidal standard* $C[G]$*-triples. If* $Q \not\subseteq Q'$*, then the* C*-vector space*

$$\mathrm{Ext}_G^1(\mathrm{I}_G(P', \sigma', Q'), \mathrm{I}_G(P, \sigma, Q))$$

is non-zero if and only if $P' = P$, $\sigma' \cong \sigma$, *and* $Q' = Q^\alpha$ *for some* $\alpha \in \Delta_Q$, *in which case it is one-dimensional and the unique (up to isomorphism) non-split extension of* $\mathrm{I}_G(P', \sigma', Q')$ *by* $\mathrm{I}_G(P, \sigma, Q)$ *is the admissible* C*-representation of* G

$$\mathrm{Ind}_{P(\sigma)^\alpha}^G(\mathrm{I}_{M(\sigma)^\alpha}(M(\sigma)^\alpha \cap P, \sigma, M(\sigma)^\alpha \cap Q)).$$

Proof. There is a natural short exact sequence of admissible C-representations of G

$$0 \to \sum_{Q' \subsetneq Q'' \subseteq P(\sigma')} \mathrm{Ind}_{Q''}^G(\sigma') \to \mathrm{Ind}_{Q'}^G(\sigma') \to \mathrm{I}_G(P', \sigma', Q') \to 0. \tag{19}$$

Note that we can restrict the sum to those Q'' that are minimal, i.e. of the form Q'_α for some $\alpha \in \Delta_{P(\sigma')} \backslash \Delta_{Q'}$. Moreover, we deduce from [AHV17, Theorem 3.2] that its cosocle is isomorphic to $\bigoplus_{\alpha \in \Delta_{P(\sigma')} \backslash \Delta_{Q'}} \mathrm{I}_G(P', \sigma', Q'_\alpha)$. Now if $Q \not\subseteq Q'$, then $\mathrm{Ord}_{\bar{Q}'}(\mathrm{I}_G(P, \sigma, Q)) = 0$ by [AHV17, Theorem 1.1 (ii) and Corollary 4.13] so that using Corollary 2, we see that the long exact sequence of Yoneda extensions obtained by applying the functor $\mathrm{Hom}_G(-, \mathrm{I}_G(P, \sigma, Q))$ to (19) yields a natural C-linear isomorphism

$$\mathrm{Ext}_G^{n-1}(\textstyle\sum_{Q' \subsetneq Q'' \subseteq P(\sigma')} \mathrm{Ind}_{Q''}^G(\sigma'), \mathrm{I}_G(P, \sigma, Q)) \xrightarrow{\sim} \mathrm{Ext}_G^n(\mathrm{I}_G(P', \sigma', Q'), \mathrm{I}_G(P, \sigma, Q))$$

for all $n \geq 1$. In particular, with $n = 1$ and using the identification of the cosocle of the sum and [AHHV17, I.3 Theorem 2], we deduce that the C-vector space in the statement is non-zero if and only if $P' = P$, $\sigma' \cong \sigma$, and $Q = Q'_\alpha$ for some $\alpha \in \Delta_{P(\sigma')} \backslash \Delta_{Q'}$ (or equivalently $Q' = Q^\alpha$ for some $\alpha \in \Delta_Q$), in which case it is one-dimensional. Finally, using again [AHV17, Theorem 3.2], we see that for all $\alpha \in \Delta_Q$ the admissible C-representation of G in the statement is a non-split extension of $\mathrm{I}_G(P, \sigma, Q^\alpha)$ by $\mathrm{I}_G(P, \sigma, Q)$. $\square$

Corollary 7. *Assume* $\mathrm{char}(F) = p$*. Let* π *and* π' *be two irreducible admissible* C*-representations of* G*. If* π *is supercuspidal and* π' *is not the extension to* G *of a supercuspidal representation of a Levi subgroup of* G*, then* $\mathrm{Ext}^1_G(\pi', \pi) = 0$*.*

Proof. By [AHHV17, I.3 Theorem 3], there exist two supercuspidal standard $C[G]$-triples (P, σ, Q) and (P', σ', Q') such that $\pi \cong \mathrm{I}_G(P, \sigma, Q)$ and $\pi' \cong \mathrm{I}_G(P', \sigma', Q')$. The assumptions on π and π' are equivalent to $P = G$ and $Q' \neq G$. In particular, $Q \not\subseteq Q'$ and $P \neq P'$ so that $\mathrm{Ext}^1_G(\pi', \pi) = 0$ by Proposition 6. □

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